
Modern approaches to quantum gravity

Homework 12 Fall 2025

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1. The bulk to boundary propagator

We will consider Eucldidean $EAdS_{d+1}$ with metric $ds^2 = \frac{dz^2+d\vec{x}^2}{z^2}$. The goal of this exercise is to the find the solution of $\Box\phi = m^2\phi$ such that $\phi(x,z) \sim \tilde{\phi}(x)z^{d-\Delta}$ + subleading, at the boundary, and ϕ is regular as $z \to \infty$.

(a) Solve $\Box\phi = m^2\phi$ in the coordinates mentionned above, using the Ansatz

$$\phi = e^{i\vec{k}\cdot\vec{x}} f_{\vec{k}}(z) \tag{1}$$

The equation of motion should reduce to

$$z^2 f'' - (d-1)zf' - \vec{k}^2 z^2 f = m^2 f \tag{2}$$

and show that the two independent solutions can be written as

$$f^{(1)}_{\vec{k}}(z) = z^{d/2} I_{\Delta-d/2}(|\vec{k}|z) \qquad f^{(2)}_{\vec{k}}(z) = z^{d/2} K_{\Delta-d/2}(|\vec{k}|z) \tag{3}$$

where I_α and K_α are the modified Bessel functions.

(b) Argue that only $f^{(2)}_{\vec{k}}$ doesn't blow up as $z \to \infty$ (interior of the spacetime) and that

$$g_{\vec{k}}(z) = \frac{2}{\Gamma(\Delta - d/2)} \left(\frac{2}{|\vec{k}|}\right)^{d/2-\Delta} z^{d/2} K_{\Delta-d/2}(|\vec{k}|z) \tag{4}$$

is the relevant normalization such that

$$\phi(\vec{x},z) = \int d^dk \tilde{\phi}'(\vec{k}) e^{i\vec{k}\cdot\vec{x}} g_{\vec{k}}(z) \to z^{d-\Delta}\tilde{\phi}(\vec{x}) \qquad (z \to 0) \tag{5}$$

where $\phi'(\vec{k})$ is the Fourier transform of $\phi(\vec{x})$.

(c) Show that

$$\int d^dk e^{i\vec{k}\cdot(\vec{x}-\vec{y})} g_{\vec{k}}(z) \propto \left(\frac{z}{z^2 + (\vec{x}-\vec{y})^2}\right)^{\Delta}. \tag{6}$$

(d) Conclude that

$$\phi(\vec{x},z) = \int d^dy \left(\frac{z}{z^2 + (\vec{x}-\vec{y})^2}\right)^{\Delta} \tilde{\phi}(y) \tag{7}$$

2. Correlation functions

(a) Consider the gravitational low energy action of string theory (in Einstein frame). One can write it schematically as

$$S \sim \frac{1}{G} \int d^{10}x \sqrt{-g} (R + c_1 \alpha' R^2 + c_2 \alpha' R_{\mu\nu} R^{\mu\nu} + \dots) \tag{8}$$

where $\alpha' = l_s^2$ controls the *higher curvature* corrections, beyond the Einstein-Hilbert action (c_i are some order one constants), and G is the 10-dimensional Newton's constant, which is expressed in terms of the Planck length l_p and string length l_s as $G = l_p^8 = g_s^2 l_s^8$. When we consider $AdS_5 \times S^5$ with radius R_{AdS} as a background around which we will expand our fields, it is useful to rescale the metric $g_{\mu\nu}$ to make it dimensionless. Argue that under this rescaling,

$$S \to \frac{R_{\text{AdS}}^8}{G} \int dx \sqrt{-g} (R + c_1 \alpha' R_{\text{AdS}}^{-2} R^2 + c_2 \alpha' R_{\text{AdS}}^{-2} R_{\mu\nu} R^{\mu\nu} + \dots) \,. \tag{9}$$

Conclude using $R_{\text{AdS}}^4 = 4\pi g_s l_s^4 N$ that higher curvature corrections are of order $1/\sqrt{g_s N} = 1/\sqrt{\lambda}$ (remember that $g_s = g_{\text{YM}}^2$). Show also that expanding around the $AdS_5 \times S^5$ metric, cubic interactions will start mattering at order $\frac{G}{R_{\text{AdS}}^8} \sim \frac{1}{N^2}$.

(b) We will now consider the limit in which $\lambda = g_{\text{YM}}^2 N = g_s N \gg 1$ and $N \gg 1$, and consider $1/N$ corrections. As you derived above, this would correspond to consider cubic interactions on top of the $AdS_5 \times S^5$ geometry. As a toy example, let us consider three scalar fields ϕ_i on AdS_5 which have masses m_i^2 and interact through a 3-pt interaction. Namely,

$$S = \int d^5x \sqrt{g} \left[\sum_i \left(\frac{1}{2} (\partial \phi_i)^2 + \frac{1}{2} m_i^2 \phi_i^2 \right) + \lambda \phi_1 \phi_2 \phi_3 \right] \tag{10}$$

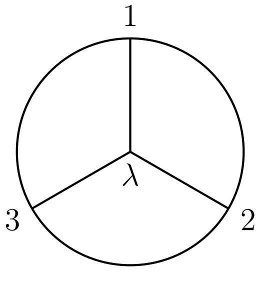
As you learned in the course, this is related to the boundary theory generating functional once evaluated on-shell with the boundary conditions set by the boundary sources. This implies that the 3-pt function on the boundary is obtained through

$$\langle \mathcal{O}_1(\vec{x}_1) \mathcal{O}_2(\vec{x}_2) \mathcal{O}_3(\vec{x}_3) \rangle = -\lambda \int d^5x \sqrt{g} K_{\Delta_1}(x; \vec{x}_1) K_{\Delta_2}(x; \vec{x}_2) K_{\Delta_3}(x; \vec{x}_3) \tag{11}$$

where x is 5-dimensional and $\vec{x}$ is 4-dimensional and

$$K_\Delta(\vec{x}, z; \vec{y}) \equiv \frac{\Gamma(\Delta)}{\pi^2 \Gamma(\Delta - 2)} \left(\frac{z}{z^2 + (\vec{x} - \vec{y})^2} \right)^\Delta \,. \tag{12}$$

This is usually depicted as a Witten diagram :



Compute this diagram and show that

$$\langle \mathcal{O}_1(\vec{x}_1)\mathcal{O}_2(\vec{x}_2)\mathcal{O}_3(\vec{x}_3)\rangle = \frac{\lambda a_1}{|\vec{x}_1-\vec{x}_2|^{\Delta_1+\Delta_2-\Delta_3}|\vec{x}_1-\vec{x}_3|^{\Delta_1+\Delta_3-\Delta_2}|\vec{x}_2-\vec{x}_3|^{\Delta_2+\Delta_3-\Delta_1}} \tag{13}$$

where

$$\begin{aligned} a_1 = -&\frac{\Gamma\left[\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)\right]\Gamma\left[\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2)\right]\Gamma\left[\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)\right]}{2\pi^4\Gamma(\Delta_1-2)\Gamma(\Delta_2-2)\Gamma(\Delta_3-2)} \\ &\cdot\Gamma\left[\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3)-2\right] \end{aligned} \tag{14}$$

(c) Now consider the 4-point function, still using the same action (10), with masses $m_i^2 = m^2$. Is it zero or nonzero, and why?

3. Two-point function of a massive scalar on the BTZ black hole

Consider the non-rotating BTZ black hole (AdS radius $\ell = 1$)

$$ds^2 = -(r^2-r_+^2)\,dt^2 + \frac{dr^2}{r^2-r_+^2} + r^2 d\phi^2, \qquad \phi \sim \phi + 2\pi, \tag{15}$$

with Hawking temperature $T = \frac{r_+}{2\pi}$. A massive scalar field Φ satisfies the Klein–Gordon equation

$$(\Box - m^2)\Phi = 0. \tag{16}$$

(a) Use the mode ansatz $\Phi(t,r,\phi) = e^{-i\omega t + ik\phi}\,R(r)$ (with $k \in \mathbb{Z}$), and derive the radial ODE for $R(r)$. Show that after a suitable change of variable (e.g. to a dimensionless coordinate that maps the horizon to 0 and the boundary to 1), the equation becomes a hypergeometric equation.

(b) Impose *ingoing* boundary conditions at the horizon and write the corresponding solution in terms of hypergeometric functions. Determine the two independent near-boundary behaviors and their coefficients (the "non-normalisable/source" and "normalisable/response" coefficients).

(c) Using the real-time AdS/CFT prescription, define the (momentum-space) retarded Green's function $G_R(\omega,k)$ of the dual operator $\mathcal{O}$ as the ratio (up to an overall ω-independent normalization) of response over source coefficients from part (b). Extract the poles of $G_R(\omega,k)$ in the lower-half ω-plane.

(d) The dual CFT operator has conformal dimension

$$\Delta = 1 + \sqrt{1+m^2}, \qquad h = \bar{h} = \frac{\Delta}{2}. \tag{17}$$

Consider the CFT at temperature T on the line, with lightcone coordinates $x^\pm = t \pm \sigma$. Starting from the plane correlator $\langle \mathcal{O}(z,\bar{z})\mathcal{O}(0)\rangle \propto z^{-2h}\bar{z}^{-2\bar{h}}$, map to the thermal correlator on the cylinder by a Weyl transformation and obtain

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_T \propto \left(\frac{\pi T}{\sinh\big(\pi T(x^+ - i\epsilon)\big)}\right)^{2h}\left(\frac{\pi T}{\sinh\big(\pi T(x^- - i\epsilon)\big)}\right)^{2\bar{h}}. \tag{18}$$

Argue that the retarded correlator has poles at the same locations as in part (c).